AN APPLICATION OF SOMMERFELD'S COMPLEX ORDER WAVE FUNCTIONS TO ANTENNA THEORY

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Abstract

In the past wave functions of integral order have been used quite advantageously in the solution of certain antenna and boundary-value problems. However, in some instances these wave functions are completely alien to the problem and introduce difficulties which, indeed, can be resolved but only at the expense of logical simplicity. To place in evidence the usefulness and "naturalness" of complex order wave functions for the solution of certain problems, we examine theoretically the input admittance of a boss antenna with the aid of these functions.

Introduction

Suppose we desire to construct a solution to the wave equation

$$(\nabla^2 + k^2)\, u = 0 \tag{1}$$

where u must satisfy the Sommerfeld radiation condition and must assume prescribed values on the surface of a sphere of radius a. Using spherical coordinates r, θ, ϕ with origin at center of sphere, assuming that u is independent of the azimuthal angle ϕ, and requiring that on the surface of sphere

$$\alpha u + \beta \frac{\partial}{\partial r} u = f(\theta) \tag{2}$$

where α and β are constants, we consider a solution to (1) of the form

$$u(r, \theta) = \sum_{n=0}^{\infty} A_n P_n(\cos\theta)\, h_n^{(1)}(kr) \tag{3}$$

where $P_n(\cos\theta)$ is the Legendre <u>polynomial</u> of <u>integral order</u> n and $h_n^{(1)}(kr)$ is the spherical Hankel function of the same order. The A_n's are constants which have to be determined by requiring that (3) satisfy (2) when

$r = a$. As it stands, (3) satisfies the radiation condition,

$$\lim_{r \to \infty} r\left(\frac{\partial}{\partial r} u - iku\right) = 0 \tag{4}$$

because as $r \to \infty$ the spherical Hankel functions have the behaviour of an outwardly propagating wave,

$$h_n^{(1)}(kr) \sim (-i)^{n+1} \frac{e^{ikr}}{kr} \tag{5}$$

To determine the unknown constants, the A_n's, we substitute (3) into (2) and use the orthogonality property of the Legendre polynomials,

$$\int_0^{\pi} P_n(\cos\theta)\, P_m(\cos\theta) \sin\theta \, d\theta = \delta_{nm} \frac{2}{2m+1} \tag{6}$$

where $\delta_{nm} = 0, 1$ when $n \neq m$, $n = m$ respectively. Thus

$$A_m = \frac{2m+1}{2} \frac{\int_0^{\pi} f(\theta)\, P_m(\cos\theta) \sin\theta \, d\theta}{\left[\alpha\, h_m^{(1)}(ka) + \beta \frac{\partial}{\partial r} h_m^{(1)}(ka)\right]} \tag{7}$$

Substituting (7) into (3) we have the desired solution. It is clear that the crux of this problem is the orthogonality of the Legendre polynomials. If this orthogonality did not exist, it would be impossible to determine the A_n's. However, for certain problems it would be much more convenient and "natural" if the radial functions, instead of the angular functions, were orthogonal. This is true, for example, if the wave function has to satisfy on a conical surface boundary conditions which depend on r. In the boss antenna problem the wave function must satisfy just such boundary conditions.

In this note we first develop some properties of the complex order wave functions. We find that u can be represented in the form of an infinite series, each term of which is the product of an angular function and a radial function. The angular and radial functions prove to be the

Legendre functions and the spherical Hankel functions of complex order respectively. And now it is the spherical Hankel functions of complex order that are orthogonal. Then we apply these wave functions to the calculation of the input impedance of a boss antenna.

Complex Order Wave Functions

Let us consider in spherical coordinates r, θ, ϕ, a TM field, $\underline{E} = \underline{r}\, E_r + \underline{\theta}\, E_\theta$, $\underline{H} = \phi\, H_\phi$ where $\underline{r}, \underline{\theta}, \underline{\phi}$, are the unit vectors and E_r, E_θ, H_ϕ are the non-vanishing components of the electromagnetic field. For steady-state, i.e., time dependence $\exp(-i\omega t)$, the complex vectors $\underline{E}$ and $\underline{H}$ satisfy the Maxwell equations,

$$\nabla \times \underline{E} = i\omega\mu \underline{H} \tag{8}$$

$$\nabla \times \underline{H} = i\omega \underline{E} \tag{9}$$

from which it follows that

$$\nabla \times \nabla \times \underline{H} = k^2\, \underline{H} \tag{10}$$

where $k^2 = \omega^2 \mathcal{E}\mu$ and $\mathcal{E}, \mu$ denote the dielectric constant and permeability of free space. If we assume that $\underline{E}$ and $\underline{H}$ are independent of the azimuthal angle ϕ, then in spherical coordinates (10) takes the form

$$\frac{\partial^2}{\partial r^2}(r\, H_\phi) + \frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(H_\phi \sin\theta)\right] + k^2 r\, H_\phi = 0 \tag{11}$$

If we set

$$H_\phi = i\omega\mathcal{E}\frac{\partial}{\partial\theta}\, u \tag{12}$$

then (11) can be written as

$$i\omega\mathcal{E}\frac{\partial}{\partial\theta}\left[(\nabla^2 + k^2)\, u(r,\theta)\right] = 0\ . \tag{13}$$

We choose $u(r,\theta)$ such that

$$(\nabla^2 + k^2)\, u(r,\theta) = 0 \tag{14}$$

for all permissible values of r and θ. Then according to (9) and (12) the electromagnetic field is given by the following expressions:

$$E_r = -\frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta \frac{\partial}{\partial\theta} u) \tag{15}$$

$$E_\theta = \frac{1}{r} \frac{\partial}{\partial\theta} \frac{\partial}{\partial r} (r u) \tag{16}$$

$$H_\phi = i\omega\mathcal{E} \frac{\partial}{\partial\theta} u \tag{17}$$

Now we examine the solution of the wave equation (14) that satisfies the radiation condition (4) and the boundary condition,

$$\frac{\partial}{\partial r} (r u) = 0 \qquad \text{when } r = a \tag{18}$$

In view of (16) it is seen that the condition (18) is equivalent to requiring that E_θ vanish over surface of sphere of radius a.

Writing $u(r,\theta)$ explicitly in separated form, $u = R(r) P(\cos\theta$ (14) yields two ordinary differential equations

$$\frac{1}{\sin\theta} \frac{d}{d\theta} (\sin\theta \frac{dP}{d\theta}) + CP = 0 \tag{19}$$

$$\frac{1}{R} \frac{d}{dr} (r^2 \frac{dR}{dr}) + k^2 r^2 = C \tag{20}$$

where C is the separation constant. If we choose $C = \nu(\nu + 1)$ where ν is unrestricted (a complex number), the solution of (19) is the Legendre function $P_\nu(\cos\theta)$. This function is identical to the hypergeometric function, i.e.,

$$P_\nu (\cos\theta) = F(-\nu, \nu + 1, 1, \frac{1 - \cos\theta}{2}) \tag{21}$$

and is finite for all θ except $\theta = \pi$. The corresponding solution of (20) is the spherical Hankel function of the first kind $h_\nu^{(1)}(kr)$ and it is related to the cylindrical Hankel function of the first kind $H_{\nu+1/2}^{(1)}(kr)$

according to the relation,

$$h_\nu^{(1)}(kr) = \sqrt{\frac{\pi}{2kr}}\; H_{\nu+1/2}^{(1)}(kr) \tag{22}$$

Hence, we can write the solution of (14) as

$$u(r,\theta) = \sum A_\nu h_\nu^{(1)}(kr)\, P_\nu(\cos\theta) \tag{23}$$

where the summation is over all values of ν determined by the boundary condition (18). Substituting (23) into (18) we obtain

$$\sum A_\nu \frac{\partial}{\partial r}\left(a\, h_\nu^{(1)}(ka)\right) P_\nu(\cos\theta) = 0 \tag{24}$$

To satisfy (24) the ν's are chosen so that

$$\left[\frac{\partial}{\partial r}\left(r\, h_\nu^{(1)}(kr)\right)\right]_{r=a} = 0 \quad . \tag{25}$$

We denote these ν's as ν_1, ν_2, ν_3, etc.

The radial functions $h_\nu^{(1)}(x)$ are orthogonal over the range $x = ka$ to $x = \infty$. To show this we recall from (20) that for any ν, say, ν_n

$$x\,\frac{d^2}{dx^2}\left(x\, h_{\nu_n}^{(1)}(x)\right) + \left(x^2 - \nu_n(\nu_n + 1)\right) h_{\nu_n}^{(1)}(x) = 0 \tag{26}$$

and for any other value of ν, say, ν_m

$$x\,\frac{d^2}{dx^2}\left(x\, h_{\nu_m}^{(1)}(x)\right) + \left(x^2 - \nu_m(\nu_m + 1)\right) h_{\nu_m}^{(1)}(x) = 0 \tag{27}$$

Multiplying (26) by $h_{\nu_m}^{(1)}$ and (27) by $h_{\nu_n}^{(1)}$ and then integrating the difference from $x = ka$ to $x = \infty$, we obtain

$$\left[\nu_m(\nu_m+1)-\nu_n(\nu_n+1)\right]\int_{ka}^{\infty} h_{\nu_n}^{(1)}(x)\, h_{\nu_m}^{(1)}(x)\, dx =$$

$$\int_{ka}^{\infty}\left[x\, h_{\nu_n}^{(1)}(x)\ \frac{d^2}{dx^2}\left(x\, h_{\nu_m}^{(1)}(x)\right) - x\, h_{\nu_m}^{(1)}(x)\ \frac{d^2}{dx^2}\left(x\, h_{\nu_n}^{(1)}(x)\right)\right] dx =$$

$$\left[x\, h_{\nu_n}^{(1)}(x)\ \frac{d}{dx}\left(x\, h_{\nu_m}^{(1)}(x)\right) - x\, h_{\nu_m}^{(1)}(x)\ \frac{d}{dx}\left(x\, h_{\nu_n}^{(1)}(x)\right)\right]_{ka}^{\infty} \qquad (28)$$

where the last equality results from an integration by parts. The integrated term disappears at $x = ka$ and $x = \infty$ by virtue of (25) and the asymptotic behaviour of $h_{\nu}^{(1)}(x)$. Thus

$$\int_{ka}^{\infty} h_{\nu_n}^{(1)}(x)\, h_{\nu_m}^{(1)}(x)\, dx = \delta_{nm}\ N_{\nu_n}(ka) \qquad (29)$$

$N_{\nu_n}(ka)$ is a normalization factor which can be obtained from (28) by an application of l'Hospital's rule for the limit $\nu_n \to \nu_m$, and, of course, δ_{nm} is the well-known Kronecker delta.

Hence, the solution of (14) which satisfies the boundary condition (18) and the Sommerfeld radiation condition is given by a sum over all ν's:

$$u(r,\theta) = \sum A_{\nu}\, h_{\nu}^{(1)}(kr)\ P_{\nu}(\cos\theta) \qquad (30)$$

where the A_{ν}'s are as yet undetermined constants. Once we are given $u(r,\theta_o)$ on any conical surface $\theta = \theta_o$, the A_{ν}'s are determinable by virtue of the orthogonality condition (29). Contrasting (3) and (30) we see that (3) is appropriate for boundary conditions on a sphere, whereas (30) is appropriate for cones.

Substituting (30) into (15), (16), and (17) we find the electromagnetic field in terms of the complex order wave functions, $h_{\nu}^{(1)}(kr)\ P_{\nu}(\cos\theta)$

$$H_{\phi}(r,\theta) = i\omega\mathcal{E}\sum A_{\nu} h_{\nu}^{(1)}(kr)\,\frac{\partial}{\partial\theta}\,P_{\nu}(\cos\theta)\quad, \tag{31}$$

$$E_{\theta}(r,\theta) = \frac{1}{r}\sum A_{\nu}\frac{\partial}{\partial r}\left(r\,h_{\nu}^{(1)}(kr)\right)\frac{\partial}{\partial\theta}\,P_{\nu}(\cos\theta), \tag{32}$$

and since $P_{\nu}(\cos\theta)$ satisfies (19) when $C = \nu(\nu+1)$,

$$E_{r}(r,\theta) = \frac{1}{r}\sum A_{\nu}\ \nu(\nu+1)\ h_{\nu}^{(1)}(kr)\ P_{\nu}(\cos\theta) \tag{33}$$

An Application to Antenna Theory

To place in evidence the usefulness of these complex order wave functions, we shall use them to compute the admittance of a boss antenna. A boss antenna consists of a coaxial line fitted with an infinite flange and a hemispherical boss at the end of the inner conductor (fig. 1). Except for the addition of a hemispherical boss, it is identical to the circular diffraction antenna.[2]

We assume the antenna is excited by a single propagating mode (the principal mode) in the coaxial region. This mode has no azimuthal variation. The non-vanishing components of the field in the coaxial region, $z \leq 0$, $a \leq \rho \leq b$, are H_{ϕ}, E_{ρ}, E_{z} and in the antenna region, $r \geq a$, $\theta \leq \frac{\pi}{2}$, the non-vanishing components are H_{ϕ}, E_{r}, E_{θ}. Cylindrical coordinates are used in the coaxial region and spherical coordinates in the antenna region.

In the antenna region, according to (33), the r-component of the electric field is given by

$$E_{r}(r,\theta) = \frac{1}{r}\sum A_{\nu}\ \nu(\nu+1)h_{\nu}^{(1)}(kr)\ P_{\nu}(\cos\theta),\qquad r \geq a,\quad \theta \leq \frac{\pi}{2}\ . \tag{33a}$$

If we denote $E_{r}(r,\frac{\pi}{2})$ by $\mathcal{E}(r)$, (33a) becomes

$$0 = \frac{1}{r}\sum A_{\nu}\ \nu(\nu+1)\ h_{\nu}^{(1)}(kr)\ P_{\nu}(0),\qquad r \geq b,\quad \theta = \frac{\pi}{2} \tag{34a}$$

$$\mathcal{E}(r) = \frac{1}{r}\sum A_{\nu}\ \nu(\nu+1)\ h_{\nu}^{(1)}(kr)\ P_{\nu}(0),\qquad a \leq r \leq b,\quad \theta = \frac{\pi}{2} \tag{34b}$$

since the component of the electric field tangent to the perfectly conducting baffle must disappear. Multiplying both sides of (34) by $rh_\nu^{(1)}(kr)$ and integrating from $kr = ka$ to $kr = \infty$, we obtain by virtue of the orthogonality relation (29),

$$A_\nu = \frac{\int_{ka}^{\infty} r\,\mathcal{E}_\rho(r)\; h_\nu(kr)\; d(kr)}{\nu(\nu+1)\; P_\nu(0)\;\; N_\nu(ka)} \tag{35}$$

Substituting (35) into (31), we see that the magnetic field at any point in the antenna region is given by

$$H_\phi^{(+)}(r,\theta) = i\,\omega\,\mathcal{E}\int_{ka}^{\infty} r'\mathcal{E}_\rho(r')\; d(kr')\sum \frac{h_\nu(kr)\; h_\nu(kr')\;\; \frac{\partial}{\partial\theta} P_\nu(\cos\theta)}{\nu(\nu+1)\; P_\nu(0)\; N_\nu(ka)} \tag{36}$$

According to eq. (2.34) of reference 2, the magnetic field in the coaxial region $a \leq \rho \leq b$, $z \leq 0$ is given by

$$H_\phi^{(-)}(\rho,z) = \frac{I(z)}{2\pi\rho} + i\,\omega\,\mathcal{E}\int_a^b \mathcal{E}_\rho(\rho')\;\rho' d\rho' \sum \frac{R_n(\rho)\; R_n(\rho')}{\sqrt{\lambda_n^2 - k^2}}\;\; e^{\sqrt{\lambda_n^2 - k^2}\; z} \tag{37}$$

where $I(z)$ is the coaxial line current and $\mathcal{E}_\rho(\rho)$ is the ρ-component of the electric field across the aperture. $\mathcal{E}_\rho(\rho)$ and $\mathcal{E}_\rho(r)$ of (34b) are identical since in the plane of the baffle the r-coordinate of the spherical coordinate system and the ρ-coordinate of the cylindrical coordinate system are identical. The eigenvalues λ_n and the eigenfunctions $R_n(\rho)$ are defined by

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\,\frac{\partial}{\partial\rho} - \frac{1}{\rho^2} + \lambda_n^2\right)\; R_n(\rho) = 0 \tag{38}$$

$$\left(\frac{\partial}{\partial\rho} + \frac{1}{\rho}\right)\; R_n(\rho) = 0 \quad \text{at} \quad \rho = a,b \tag{39}$$

and the λ_n's are roots of the transcendental equation

$$J_o(\lambda_n a)\, N_o(\lambda_n b) = N_o(\lambda_n a)\, J_o(\lambda_n b) \tag{40}$$

The orthogonality relations,

$$\int_a^b R_n(\rho)\; R_m(\rho)\; \rho\, d\rho = \delta_{nm} \tag{41}$$

$$\int_a^b R_n(\rho)\, d\rho = 0 \tag{42}$$

immediately follow from (38) and (39).

Since H_ϕ must be continuous across the aperture, we have $H_\phi^{(+)}(r,\frac{\pi}{2}) = H_\phi^{(-)}(\rho,0)$ for $a \leq \rho \leq b$. Thus from (36) and (37)

$$\frac{I(0)}{2\pi\rho} + i\omega\mathcal{E}\int_a^b \mathcal{E}(\rho')\,\rho'\,d\rho' \sum \frac{R_n(\rho)\,R_n(\rho')}{\sqrt{\lambda_n^2 - k^2}} =$$

$$i\omega\mathcal{E}k \int_{ka}^{\infty} r'\,\mathcal{E}(r')\,dr' \sum \frac{h_\nu(kr)\,h_\nu(kr')\,\frac{\partial}{\partial\theta}P_\nu(0)}{\nu(\nu+1)\,P_\nu(0)\,N_\nu(ka)} \tag{43}$$

This is an integral equation. We shall not attempt to solve it. Rather, we shall use it to formulate a variational principle for the antenna admittance.

From (2.26) and (2.27) of reference 2 the coaxial line current and voltage are given by

$$I(z) = 2\pi\,(\alpha\, e^{ikz} + \beta e^{ikz}) \tag{44}$$

$$V(z) = \int_a^b E\rho^{(-)}(\rho,z)\,d\rho = \sqrt{\frac{\mu}{\mathcal{E}}}\,(\alpha\, e^{ikz} - \beta e^{ikz})\,\log\frac{b}{a} \tag{45}$$

where α and β are constants.

The characteristic admittance of the line is given by

$$Y_o = \frac{2\pi}{\sqrt{\frac{\mu}{\varepsilon}}\log(\frac{b}{a})} \tag{46}$$

Now let us derive the variational principle for the admittance $Y(0)$ where

$$Y(0) = \frac{I(0)}{V(0)} \tag{47}$$

We do this by multiplying (43) by $\rho\,\xi(\rho)$ and integrating from $\rho = a$ to $\rho = b$, and finally dividing the resultant equation by

$$\left[\int_a^b \xi(\rho)\, d\rho\right]^2 = \left[V(0)\right]^2$$

Thus

$$\frac{1}{2\pi}\, Y(0) + \frac{i\,\omega\,\varepsilon}{\left[\int_a^b \xi(\rho) d\rho\right]^2} \sum \frac{\left(\int_a^b \rho\, \xi(\rho)\, R_n(\rho)\, d\rho\right)^2}{\sqrt{\lambda_n^2 - k^2}} =$$

$$\frac{i\,\omega\,\varepsilon\, k}{\left[\int_a^b \xi(\rho) d\rho\right]^2} \sum \frac{\left(\int_a^b \rho' d\rho'\, \xi(\rho') h_{\nu_n}^{(1)}(k\rho')\right)^2}{\nu(\nu+1)\; N_\nu(ka)} \; \frac{\frac{\partial}{\partial\theta} P\,(0)}{P_\nu(0)} \tag{48}$$

It can be easily shown that $Y(0)$ is <u>stationary</u> with respect to small variations of $\xi(\rho)$ about the true $\xi(\rho)$ determined by the integral equation (43).

We choose $\frac{1}{\rho}$ as a trial function. That is, we let $\xi(\rho) = \frac{1}{\rho}$ in (48). Due to the orthogonality relation (42), (48) becomes

$$Y(0) = \frac{2\pi\, i\,\omega\,\varepsilon\, k}{\left[\log\frac{b}{a}\right]^2} \sum \frac{\left(\int_a^b h_\nu^{(1)}(k\rho)\, d\rho\right)^2}{\nu(\nu+1)\; N_\nu(ka)} \; \frac{\frac{\partial}{\partial\theta} P_\nu(0)}{P_\nu(0)} \tag{49}$$

This is an accurate expression for the admittance of the antenna. It is sometimes more useful to consider the ratio $Y(0)/Y_o$, which is a dimensionless quantity.

$$\frac{Y(0)}{Y_o} = \frac{i}{\log\left(\frac{b}{a}\right)} \frac{\left(\int_{ka}^{kb} h_{\nu}^{(1)}(k\rho)\, d(k\rho)\right)^2}{\int_{ka}^{\infty}\left[h_{\nu}^{(1)}(k\rho)\right]^2 d(k\rho)} \frac{1}{\nu(\nu+1)} \frac{\frac{\partial}{\partial\theta} P_{\nu}(0)}{P_{\nu}(0)} \tag{50}$$

This is only a formal solution, however. Since the values of the functions involved have not been tabulated, only certain limiting values[3] can be at present obtained for (50).

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2. H. Levine and C. H. Papas, J. App. Phys., Vol.22, No.1, 29-43

3. H. Bremmer, "Terrestrial Radio Waves", Elsevier, New York 1949.

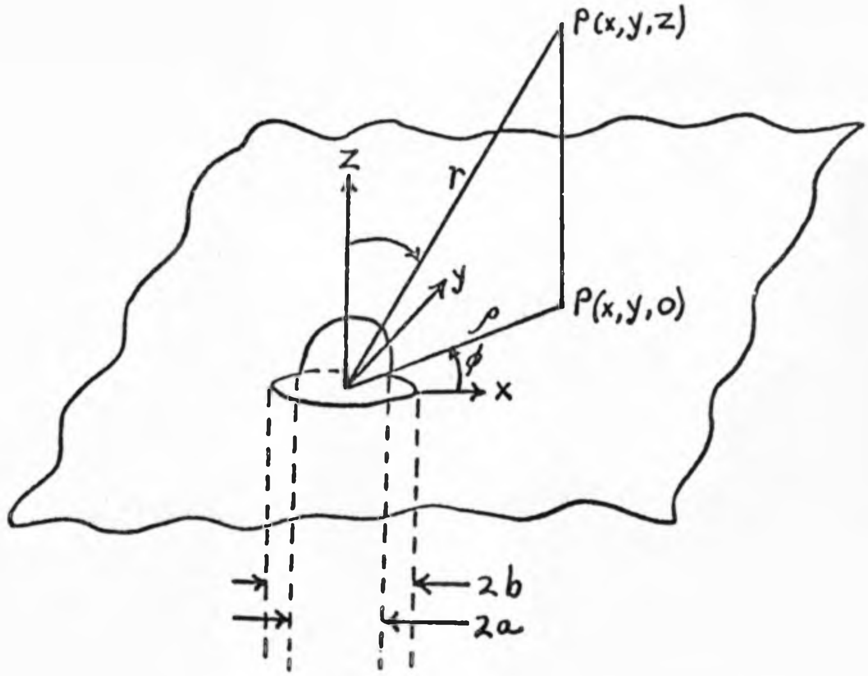

Figure 1. BOSS ANTENNA CONSISTING OF COAXIAL LINE FITTED WITH FLANGE AND HEMISPHERICAL BOSS